bombardment with alpha-particles from Ra C and Th C′. They ascribe these to the formation of K^{40} from Cl^{37} thus:

$$Cl^{37}+He^4\rightarrow K^{40}+n^1.$$

In view of the present results, however, it seems highly probable that they were detecting the formation of both K^{38} and K^{40}.

(b) The formation of K^{38} from calcium

In a previous paper[3] results obtained by one of us in a study of the radioactivities induced in calcium by deuteron bombardment were reported. It was noted that a weak activity was observed in the potassium fraction separated chemically from the irradiated metal. It was suggested that this might be due to contamination, though it was thought, in view of the fact that the half-period did not agree with that of any well-known contaminant, that it might be due to K^{38} formed thus:

$$Ca^{40}+H^2\rightarrow K^{38}+He^4;\quad K^{38}\rightarrow A^{38}+e^+.$$

Following the production of K^{38} by bombarding chlorine with alpha-particles, a search was made for this isotope in irradiated calcium. It has a sufficiently short half-life to have been unobservable in the previous experiments on account of the time needed for the chemical separation adopted.

Calcium metal was, therefore, bombarded with deuterons for half an hour and, following solution in HCl and the addition of inactive KCl, potassium was precipitated by means of perchloric acid and ethyl alcohol. In consequence the precipitate was contaminated with radioactive scandium.[3] However, on correcting for this it was found that K^{38} was present, the decay curve corresponding to a half-period of 7.6 ± 0.2 minutes. This curve is reproduced in Fig. 5.

The decay of this precipitate was measured until its corrected intensity was less than the natural leak of the electroscope, but no evidence was obtained of the 12.4 hour period of K^{42}. The expected reaction

$$Ca^{44}+H^2\rightarrow K^{42}+He^4;\quad K^{42}\rightarrow Ca^{42}+e^-$$

would thus appear to be rather improbable.

V. Conclusion

In conclusion we wish to thank the staff of the Radiation Laboratory for their cooperation, and especially Professor E. O. Lawrence for his interest and encouragement. The investigation has been aided by grants to the laboratory from the Research Corporation, the Chemical Foundation and the Josiah Macy, Jr. Foundation.

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The Disintegration of High Energy Protons

G. Nordheim, L. W. Nordheim,
Purdue University, Lafayette, Indiana

J. R. Oppenheimer and R. Serber
University of California, Berkeley, California

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The coupling between light and heavy particles assumed in the Fermi theory of β-decay makes it possible for high energy protons in passing through matter to transfer a considerable fraction of their energy to electrons and neutrinos. If we suppose that this coupling is a maximum for relative energies of the light and heavy particles of the order $\hbar c/R$, with R the range of nuclear forces, and is small for much higher relative energies, the most important process which occurs, for sufficiently energetic protons, can be pictured as a sort of photodisintegration of the proton by the contracted Coulomb field of a passing nucleus, the proton changing into a neutron and emitting a positron and a neutrino. With a coupling of the type described, and of the magnitude required by the proton-neutron forces, processes involving more than one pair of light particles will be relatively rare. The cross section for the disintegration of a proton of energy E is found to be of the order

$$2\pi(\hbar/Mc)RZ^2\alpha^2\ln^2(E/Mc^2),$$

and is very small, even for heavy nuclei. The mean energy given to the positron per disintegration is of the order

$$2(\hbar c/R)(E/Mc^2)/\ln(E/Mc^2).$$

The positrons emitted in these disintegrations can account in order of magnitude for the incidence of showers observed under thick absorbers.

1. Some Consequences of the Fermi Coupling

Until recently the only mechanisms known by which a proton could lose an appreciable fraction of its energy to electrons and γ-rays were elastic impacts of the proton with extranuclear electrons, and the relatively insignificant radiation emitted by the protons deflected in nuclear fields. The Fermi theory of β-decay provides a new mechanism for such energy transfer, because of the coupling assumed in this theory between the heavy particles (proton and neutron) and the electron-neutrino field. Owing to the weakness of the coupling necessary to explain the long lifetimes of the β-radioactive substances, this effect would be extremely small if one were to take over unmodified the coupling used in the theory of β-disintegration. However, a strong increase of the coupling with the energies of the particles concerned is suggested when one attempts an explanation of nuclear forces on the basis of Fermi's theory. The question of the consequences of this modified Fermi coupling for the behavior of heavy particles of high energy thus needs investigation.

An unambiguous answer to this question depends upon a satisfactory formulation of the problem of nuclear forces. The unsatisfactory character of present theories manifests itself in the occurrence of divergences in the method involved in their formulation, the method of successive approximations. This divergence of the method of successive approximations indicates that processes which from the point of view of this approximation are of high order, and which involve the cooperation of a large number of light particles, may be of dominant importance. The point of view adopted by Heisenberg[1] in his theory of showers would seem to be that this feature of the present inadequate field theory will also be characteristic of a correct theory, and that just those implications of present theory which would at first seem most subject to suspicion can have a qualitative validity.

Quite different from Heisenberg's suggestion is the more usual procedure of avoiding divergences by the formal device of reducing the coupling between heavy and light particles for high relative energies. In this way one can account for the finite range of nuclear forces by taking a coupling which becomes small for electrons and neutrinos whose de Broglie wave-length is smaller than this range ($\sim 2\times 10^{-13}$ cm). When the constants of the coupling are adjusted to give not only the range but the magnitude of nuclear forces, it turns out that even at its maximum the coupling is rather small, so that the probability of finding an electron-neutrino pair in the neighborhood of a heavy particle is less than one in ten, and the probability of finding many such pairs is negligible. In fact, if one applies the "Born approximation," and regards the coupling as small, the parameter which determines the relative magnitude of successive terms in this approximation is $\hbar/McR$, where R is the range of nuclear forces.[2] This model is now so much modified as compared to that of Heisenberg that it no longer affords any explanation of showers.

This model makes it possible to give a quantitative estimate of the probability that a proton will disintegrate in its passage through matter into a neutron, a positron, and a neutrino, and of the magnitude of the energy loss to light particles for which the Fermi coupling is responsible. For very high energies such a disintegration of the proton will occur for quite distant impacts with atomic nuclei. These disintegrations may be thought of as a sort of photo-effect of the proton by the contracted Coulomb field of the passing nucleus. For sufficiently high energies these distant impacts, which are relatively easy to discuss, will contribute the dominant terms to the probability of proton disintegration.[3]

[1] W. Heisenberg, Zeits. f. Physik **101**, 533 (1936).

[2] Against the correctness of this formulation is the equality of the forces between proton and neutron and between proton and proton, since these appear in different orders in the Born approximation. It would seem that the electron-neutrino theory of nuclear forces could only very artificially be made to explain this equality, as well as the range and magnitude of the forces. The generalization of the β-transformation theory proposed by G. Gamow and E. Teller, Phys. Rev. **51**, 289 (1937), would give the same order of magnitude for the proton disintegration effect as derived in this paper.

[3] For the range of energies actually found in the cosmic rays, the probability of disintegration through a direct impact of the proton and nucleus may be expected, on the basis of our model, to have a magnitude comparable to that of the process we discuss for light elements, but to be considerably smaller for heavy ones. In comparing the results of our calculation with the observed incidence of sea level showers, it should not be forgotten that we have not taken these direct impacts into account, and that their contribution may be sensitive to the model assumed.

2. The Form of the Coupling

We investigate first the possibilities of formulating a modified and "cut-off" Fermi coupling convenient for the treatment both of nuclear forces and of high energy disintegrations.

The simplest form of the Fermi interaction energy is[4]

$$H_{PN}=G(mc^2)(\hbar/mc)^3\int(\psi_N{}^*\beta\varphi_e)(\varphi_\nu{}^*\beta\psi_P)d\tau, \quad (1)$$

which describes the transition of a proton, P, into a neutron, N, with the emission of a positron, e, and a neutrino, ν. Here the matrix β is the coefficient of the mass term in the Dirac equation written in its normal form, and the integrand is thus built up of the simplest scalars which can be formed from the Dirac wave functions of the heavy and light particles. G is a pure number which fixes the magnitude of the interaction. A coupling of the type (1) will lead to an interaction between a neutron and a proton which, to the second order in G, is represented by the Majorana potential $J(r)P^M$, given by

$$[J(r)P^M]_{\text{Av}}=-\sum\{H_{NP}{}^{(1)}H_{PN}{}^{(2)}+H_{PN}{}^{(2)}H_{NP}{}^{(1)}\}/(\epsilon_e+\epsilon_\nu), \quad (2)$$

where the sum extends over all possible electron and neutrino states, with energies ϵ_e and ϵ_ν, respectively, and the upper indices (1), (2) distinguish the heavy particles.

It is well known, however, that the coupling (1), with G determined to give the correct order of magnitude for the lifetimes of the β-active nuclei ($G\sim10^{-12}$), gives the result that $J(r)$ is far too small for separations of the heavy particles of the order 10^{-13} cm, and is highly singular, behaving like r^{-5}, as $r\to0$. In order to obtain a potential of finite range and depth, and proper magnitude, without violating the facts of β-decay, it is necessary to modify (1) so that:

1. The order of magnitude of H_{PN} remains unchanged for energies of the light particles in the β-decay region ($<25\ mc^2$).

2. H_{PN} increases enormously for light particle energies of the order $\hbar c/R\sim137\ mc^2$.

3. $H_{PN}\to0$ for still higher light particle energies.

[4] The form of the spin dependence in (1) has been chosen because it gives Majorana forces.

To obtain such a dependence of (1) on the energies we could, for free particles, simply multiply the integrand by a suitable amplitude factor f, which is an invariant function of the four-vector momenta of all the particles. For arbitrary states the interaction could then be found by superposition of the individual Fourier components. Since $\hbar/R\ll Mc$ we can choose a coordinate system in which the heavy particle velocities are small. In this system our invariant, f, must reduce to an arbitrary function of the energies of electron and neutrino and of the angle between their directions of motion.

Thus our matrix element for the coupling becomes, in the rest system of the heavy particle (using plane waves for the electron and neutrino),

$$H_{PN}=G(mc^2)(\hbar/mc)^3\int(\psi_N{}^*\beta u)\times(w^*\beta\psi_P)f(\mathbf{p},\mathbf{q})\exp[(i/\hbar c)(\mathbf{p}-\mathbf{q})\cdot\mathbf{r}]d\tau, \quad (3)$$

where u and w are the Dirac amplitudes for the electron and neutrino wave functions, and $\mathbf{p}$ and $\mathbf{q}$ are c times their momenta.

For $J(r)$ we obtain from (2) and (3), after carrying out the summations over the spin variables in the usual way,[5]

$$J(r)=-\frac{G^2}{(mc^2)^4}\frac{1}{2(2\pi)^6}\int\frac{|f(\mathbf{p},\mathbf{q})|^2}{\epsilon_e+q}\exp[(i/\hbar c)(\mathbf{p}-\mathbf{q})\cdot\mathbf{r}]\times d\mathbf{p}d\mathbf{q},$$

where $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$ is the distance between the two heavy particles. We can carry out the integrations over the directions of $\mathbf{p}$ and $\mathbf{q}$ if we make the assumption that there is no coupling between the directions of emission of the electron and neutrino. Neglecting the difference between p and ϵ_e, we find

$$J(r)=-\frac{G^2}{(mc^2)^4}\frac{1}{8\pi^4}\int\int\frac{|f(p,q)|^2}{p+q}\frac{\hbar c}{pr}\sin\frac{pr}{\hbar c}\times\frac{\hbar c}{qr}\sin\frac{qr}{\hbar c}p^2dpq^2dq. \quad (4)$$

The amplitude function $f(p,q)$ has to be adjusted so that (4) gives the right magnitude and range for the nuclear forces. The former will be given correctly when the energy of the first stationary state of the deuteron is ~0. Because

[5] In the first term of (2) (emission of an electron and an antineutrino by the neutron and their reabsorption by the proton) the summation is extended over all positive energy states of the electron and negative energy states of the neutrino, in the second term (emission of a positron and a neutrino by the proton and their reabsorption by the neutron) over all negative energy states of the electron and positive energy states of the neutrino. The density of electron and neutrino states in our units is $d\mathbf{p}d\mathbf{q}/(2\pi\hbar c)^6$.

of the short range, this condition is approximately satisfied when

$$\int_0^\infty J(r)rdr\sim -\hbar^2/M, \quad (5)$$

as Bethe has observed.[6] An alternative expression of this condition is the phase integral

$$\int_0^\infty [-MJ(r)]^{\frac{1}{2}}dr=\tfrac{1}{2}\pi\hbar. \quad (5a)$$

If we introduce (4) into (5) and reverse the order of integration, the integral over $a=r/\hbar c$ gives

$$\int_0^\infty a^{-1}da \sin ap \sin aq = -\tfrac{1}{2}\ln|p-q|/(p+q),$$

and the magnitude of G is determined by

$$\frac{1}{Mc^2}=\frac{G^2}{(mc^2)^4}\frac{1}{(2\pi)^4}\int\int\frac{|f(p,q)|^2}{p+q}\ln\frac{p+q}{|p-q|}pdpqdq. \quad (6)$$

The problem of finding a function f which satisfies the requirements 1, 2, and 3 has been studied by Camp,[7] and may be simply solved by taking

$$f(p,q)=(p+q)^{\frac{1}{2}}\exp[-(\alpha^2/2)|\mathbf{p}-\mathbf{q}|^2]\varphi(\mathbf{p}+\mathbf{q}),$$

which leads to the Gaussian potential

$$J(r)=-J_0e^{-r^2/(2\alpha\hbar c)^2},$$

$$J_0=\frac{G^2}{(2mc^2)^4}\frac{\pi^{\frac{3}{2}}}{(2\pi)^6}\frac{1}{\hbar c\alpha^3}\int|\varphi(\mathbf{s})|^2d\mathbf{s}=\frac{\pi}{2Mc^2\alpha^2}$$

by (5a). When this coupling is applied to the problem of proton disintegration, it leads, because of the dependence of f on the angle between $\mathbf{p}$ and $\mathbf{q}$, to unnecessarily cumbersome integrals. Since it is unlikely that any real theoretical significance is to be attached to this f, we have chosen instead a form which is convenient for calculation, and which satisfies conditions 2 and 3:[8]

$$f(p,q)=(mc^2)^{-s}p^ne^{-p/A}q^{s-n}e^{-q/A}. \quad (7)$$

If in this formula s and n are adjusted ($s\sim 12$) to give a long enough life for the high energy β-emitters Li^8 and B^{12}, the potential $J(r)$ loses all resemblance to a simple trough, and shows marked oscillations. Since we can see that our results depend little on n and s, we shall choose, when it is necessary to fix them, the "compromise" values $s=4$, $n=2$. To the question of the effect of a dependence of f on the angle between $\mathbf{p}$ and $\mathbf{q}$, we shall return later.

One can show that with (7)

$$J(r)\rightarrow\begin{cases} r^{-(2s+5)}, & r\rightarrow\infty \\ \text{const}, & r\rightarrow 0.\end{cases}$$

The relation (6) becomes

$$G^2=\frac{2^{2s+6}\pi^4}{T_s(2s+2)!}\frac{m}{M}\left(\frac{2mc^2}{A}\right)^{2s+3}, \quad (8)$$

where T_s is a number of order unity, depending on the choice of s and n. For $s=2$, $n=1$, $T_s=0.77$, and for $s=4$, $n=2$, $T_s=0.70$. As a measure of the range we can use

$$R=2\int_0^\infty J(r)rdr\Big/\int_0^\infty J(r)dr.$$

With $s=4$, $n=2$ we get $R=\frac{2}{3}\hbar c/p_0$, where $p_0=2A$ is the energy at which f has its maximum. As $R\sim 2.2\times 10^{-13}$ cm, we find in this case $p_0\sim 115\ mc^2$.

The choice (7) for f gives the simplest extrapolation of the form of interaction used in the theory of β-disintegration. Another form, which expresses the feature that f has a pronounced maximum, is the Gaussian function

$$f(p,q)=[(p+q)/mc^2]^{\frac{1}{2}}e^{-(p-p_0)^2/2\Delta^2}e^{-(q-p_0)^2/2\Delta^2}. \quad (9)$$

(9) can be considered as the limiting form of (7) for $n\sim\frac{1}{2}s\rightarrow\infty$, with the correspondence $p_0=nA$, $\Delta=n^{\frac{1}{2}}A$. The factor $[(p+q)/mc^2]^{\frac{1}{2}}$ is introduced to simplify the integration, which gives

$$J(r)=-\frac{G^2}{(mc^2)^5}\frac{\Delta^2p_0^4}{(2\pi)^3}\left(\frac{\hbar c}{p_0r}\right)^2\sin^2\frac{p_0r}{\hbar c}e^{-\Delta^2r^2/2(\hbar c)^2},$$

while the phase integral (5a) gives, for $\Delta\ll p$,

$$G^2=(2\pi)^3(m/M)(mc^2)^4/\Delta^2p_0^2. \quad (10)$$

3. Calculation of the Cross Section

The process of the disintegration of a high energy proton by the Coulomb field of a nucleus can be thought of, in the coordinate system in which the proton is initially at rest, as the emission by the proton of a positron and a neutrino, the positron then being deflected by the contracted Coulomb field of the passing

[6] Bethe and Bacher, Rev. Mod. Phys. **8**, 109 (1936).
[7] G. Camp, Phys. Rev. **51**, 1046 (1937).
[8] In the β-decay region this is equivalent to the general formulation of the coupling discussed by Uhlenbeck and Konopinski, Phys. Rev. **48**, 7 (1935).

nucleus.[9] From this formulation we can see that it is only when frequencies of the order $p_0/\hbar$ are important in this contracted field that the disintegration will be probable. In the whole of the calculation we shall confine ourselves to the case where the field is strongly contracted, where, therefore, the proton has a velocity, v, very close to that of light, and an energy large compared to its rest energy. We shall keep only the dominant terms in ξ and $\ln \xi$, where $\xi = 1/(1-v^2/c^2)^{\frac{1}{2}}$.

One way which suggests itself for this calculation is the impact parameter method of v. Weizsäcker and Williams,[10] which replaces the contracted Coulomb field by a bundle of parallel γ-rays moving in the same direction as the nucleus. This method is valid if the momentum of the field normal to the direction of motion of the nucleus can be neglected compared to the transverse momenta of the wave packet representing the positron. As we shall see, in our problem values of the impact parameter for which this condition is not satisfied give an essential contribution to the cross section.

Instead, we shall make a Fourier analysis of the Coulomb field of the passing nucleus, without neglecting the transverse component of the propagation vector of the electromagnetic field. The field of a point charge Ze moving with velocity v in the z direction is given by

$$\varphi = Ze/[z^2 + (x^2+y^2)/\xi^2]^{\frac{1}{2}},$$
$$A_z = (v/c)\varphi, \quad A_x = A_y = 0.$$

Neglecting terms of order $1/\xi^2$, we find

$$\varphi = A_z = (Ze/2\pi^2\hbar c)\int \varphi(\mathbf{K}) \exp\left[(i/\hbar c)(\mathbf{K}\cdot\mathbf{r} - kct)\right] d\mathbf{K}, \quad \varphi(\mathbf{K}) = 1/(g^2 + k^2/\xi^2),$$

where the vector $\mathbf{k}$ is the component of $\mathbf{K}$ parallel to the z axis, and $\mathbf{g}$ is the component of $\mathbf{K}$ perpendicular to the z axis. By adding the gauge terms $\varphi'(\mathbf{K}) = -\varphi(\mathbf{K})$, $\mathbf{A}'(\mathbf{K}) = -(\mathbf{K}/k)\varphi(\mathbf{K})$ we obtain $\varphi = 0$ and

$$\mathbf{A} = \int \mathbf{A}(\mathbf{K}) \exp\left[(i/\hbar c)(\mathbf{K}\cdot\mathbf{r} - kct)\right] d\mathbf{K}/(2\pi\hbar c)^3, \quad \mathbf{A}(\mathbf{K}) = -4\pi\hbar^2c^2 Ze(\mathbf{g}/k)/(g^2+k^2/\xi^2). \tag{11}$$

The matrix element of the interaction energy, $H = e(\boldsymbol{\alpha}\cdot\mathbf{A})$, of an electron and the electromagnetic field (11), between a state of momentum $\mathbf{p}'/c$ and a negative energy state of momentum $-\mathbf{p}/c$, is

$$H_{p'p} = -4\pi\hbar^2c^2Z^2e^2(g/k)(u'^*\alpha_g u)/(g^2+k^2/\xi^2), \tag{12}$$

where $\mathbf{p}'+\mathbf{p} = \mathbf{K}$, and where we write s_g for the component of a vector $\mathbf{s}$ in the direction of $\mathbf{g}$.

For the transition from the initial state (proton with momentum $\mathbf{P} = 0$) to the final state (neutron + positron + neutrino with respective momenta $\mathbf{N}$, $\mathbf{p}/c$, $\mathbf{q}/c$) through an intermediate state in which the electron has a momentum $\mathbf{p}'/c$, we obtain from (3) and (12), with the neglect of the kinetic energy of the heavy particle, the differential cross section

$$d\phi = \frac{2\pi}{\hbar c}(4\pi\hbar^2c^2)^2(mc^2)^2\left(\frac{\hbar}{mc}\right)^6 G^2Z^2e^4\frac{g^2}{k^2}\frac{|f(p',q)|^2}{(g^2+k^2/\xi^2)^2}\frac{d\mathbf{K}}{(2\pi\hbar c)^3}$$
$$\times d\rho_e d\rho_\nu \sum_w^2 |w^*\beta\psi_P|^2 \sum_{u,\,\psi_N}^2 \left| \sum_{u'}^4 \frac{(\psi_N^*\beta u')(u'^*\alpha_g u)}{E'-E_\nu}\right|^2,$$

with the density factors $d\rho_e = d\mathbf{p}/(2\pi\hbar c)^3$, $d\rho_\nu = q^2 d\Omega_q/(2\pi\hbar c)^3$, $d\Omega_q$ the element of solid angle for the directions of emission of the neutrino, and ψ_P, ψ_N, the Dirac amplitudes for the heavy particle wave

[9] The two other processes, (1) the proton is first scattered by the nucleus into an intermediate state and then emits the light particles and (2) a β-transformation of the nucleus is induced by the Coulomb field of the passing proton, are much less important than the one discussed in the text.

[10] C. F. v. Weizsäcker, Zeits. f. Physik **88**, 612 (1934); E. J. Williams, Phys. Rev. **45**, 729 (1934), Danske Vid. Selsk. Math. Fys. **13**, 4 (1935).

functions. The momentum and energy relations are

$$\mathbf{p}+\mathbf{p}'=\mathbf{K}=\mathbf{k}+\mathbf{g},$$
$$\epsilon+q=k, \tag{13}$$
$$\epsilon=[p^2+(mc^2)^2]^{\frac{1}{2}}, \quad \epsilon'=[p'^2+(mc^2)^2]^{\frac{1}{2}}.$$

Carrying out the summation over the spin directions in the usual way, and integrating over the directions of emission of the neutrino and the azimuth of $\mathbf{g}$, we obtain for the cross section

$$\phi=\frac{G^2Z^2r_0^2}{4\pi^4(mc^2)^2}\int\frac{dk}{k}\int\frac{g^3dg}{(g^2+k^2/\xi^2)^2}\int\frac{q^2|f(p'q)|^2}{k(\epsilon'^2-q^2)^2}\frac{S}{\epsilon}d\mathbf{p}, \tag{14}$$

with

$$S=\epsilon(\epsilon'^2+q^2)-2(q-mc^2)[(\mathbf{p}\cdot\mathbf{p}')+(mc^2)^2+2p_gp_g']-mc^2(\epsilon'^2-q^2-2\epsilon q),$$

and $r_0=e^2/mc^2$.

The determining factors in (14) are the amplitude function f, which we assume to be large only for values of p' and q of the order $\hbar c/R$, and the resonance denominator (ϵ'^2-q^2). We introduce polar coordinates for $\mathbf{p}$, with θ the angle between $\mathbf{p}$ and z, and φ that between the $p-z$ and $g-z$ planes, and obtain from (13)

$$\epsilon'^2-q^2=p'^2-q^2+(mc^2)^2=g^2+2k\epsilon-2p(k\cos\theta+g\sin\theta\cos\varphi).$$

For $p\gg mc$, this becomes

$$g^2+(k/p)(mc^2)^2+2kp(1-\cos\theta)-2pg\sin\theta\cos\varphi, \tag{15}$$

and the minimum value of the resonance denominator is found to be

$$(1-p/k)\{g^2+(mc^2)^2/[(p/k)(1-p/k)]\}. \tag{16}$$

From (15) and (16) we see that the principal contribution comes from values of g small compared to k. This makes it possible to simplify S by retaining only the lowest power in g.

With the help of the relation

$$2(\mathbf{p}\cdot\mathbf{p}')=q^2+2\epsilon q-\epsilon'^2+2(mc^2)^2,$$

which follows from (13) for small g, we get for the leading term of S

$$S=(\epsilon'^2-q^2)(\epsilon+q-2mc^2)-4(mc^2)^2(q-mc^2). \tag{17}$$

On the other hand, the factor $g^3/(g^2+k^2/\xi^2)^2$ shows that only $g>k/\xi$ is effective. Therefore for $k/\xi>mc^2/[(p/k)(1-p/k)]^{\frac{1}{2}}$, i.e., not too large ξ, g will determine the magnitude of the resonance denominator and we can neglect all the mass terms and put $\epsilon=p$, $\epsilon'=p'$, obtaining

$$\phi=\frac{G^2Z^2r_0^2}{4\pi^4(mc^2)^2}\int_0^\infty\frac{dk}{k}\int_{k/\xi}^\infty\frac{dg}{g}I(g),$$
$$I(g)=\int\frac{q^2|f(p',q)|^2}{p(p'^2-q^2)}d\mathbf{p}. \tag{18}$$

If we had used the v. Weizsäcker-Williams method, introducing an impact parameter ρ, and replacing the field by an approximately equivalent light quantum field, g would not, of course, have appeared in the conservation laws (13), and thus the third integral in (14) would have been independent of g. Instead of (18) we should then have found

$$\phi\sim\frac{G^2Z^2r_0^2}{4\pi^4(mc^2)^2}\int\frac{dk}{k}\int^{\hbar c\xi/k}\frac{d\rho}{\rho}I(0). \tag{18a}$$

With the correlation $\rho\sim\hbar c/g$, this agrees with (18) for $g\ll 2mc^2$, the minimum value of $mc^2/[(p/k)(1-p/k)]^{\frac{1}{2}}$, or $\rho\gg\frac{1}{2}\hbar/mc$, since it follows from (16) that under these circumstances $I(g)\sim I(0)$. Since, however, values of g larger than 2 mc^2 contribute essentially to the result, the introduction into (18a) of $\frac{1}{2}\hbar/mc$ as a lower limit of the impact parameter would be wrong. The method of the impact parameter is inapplicable.

Owing to the resonance denominator only $p'\sim q$ will contribute to the dominant term in $\ln\xi$, and in (18) we can replace p' by q in $f(p', q)$ and integrate over the directions of **p**,

$$\int d\mathbf{p}/p(p'^2-q^2)=\int Ypdp,$$

$$Y=\int_0^\pi\int_0^{2\pi}\frac{\sin\theta d\theta d\varphi}{g^2+2kp(1-\cos\theta)-2pg\sin\theta\cos\varphi}=2\pi\int_0^\pi\frac{\sin\theta d\theta}{\{[g^2+2kp(1-\cos\theta)]^2-4p^2g^2\sin^2\theta\}^{\frac{1}{2}}}. \quad (19)$$

If we introduce $x=(1-\cos\theta)$ and replace $\sin^2\theta$ by $2x$ in the second term of the denominator, we get

$$Y\sim\begin{cases}(\pi/kp)\ln[4kp/g^2(1-p/k)], & g<g_{\max}=[4kp/(1-p/k)]^{\frac{1}{2}},\\ 0 & , \ g>g_{\max}.\end{cases}$$

The integration over g gives

$$\int_{k/\xi}^{g_{\max}}g^{-1}dg\ln[4kp/g^2(1-p/k)]=\ln^2[2\xi/(k/p-1)^{\frac{1}{2}}], \quad (20)$$

and (18) becomes, using $k=p+q$,

$$\phi=\frac{G^2Z^2r_0^2}{4\pi^3(mc^2)^2}\int_0^\infty q^2|f(q,q)|^2dq\int_0^\infty\frac{dp}{(p+q)^2}\ln^2\left[\frac{2\xi}{(k/p-1)^{\frac{1}{2}}}\right].$$

In the argument of the logarithm we may take $p\sim q\sim\frac{1}{2}k$, and we obtain for the dominant term in $\ln\xi$ in the cross section[11]

$$\phi=\frac{G^2Z^2r_0^2}{4\pi^3(mc^2)^2}\ln^2\xi\int_0^\infty q|f(q,q)|^2dq.$$

For large ξ the neglect of the mass term in the resonance denominator (ϵ'^2-q^2) is no longer allowed (although the mass dependent terms in (17) always give a negligible contribution). The only change introduced by the mass term will be that we have to replace (19) by

$$Y=2\pi\int_0^2\frac{dx}{\{[g^2+(k/p)(mc^2)^2+2kpx]^2-8p^2g^2x\}^{\frac{1}{2}}}\sim\begin{cases}\dfrac{\pi}{kp}\ln\dfrac{4kp}{(1-p/k)g^2+\frac{1}{2}(k/p)(mc^2)^2}, & g<g_{\max},\\ 0 & , \ g>g_{\max},\end{cases}$$

where $g_{\max}$ is the value of g for which the argument of the logarithm is unity, and instead of (20) we will have

$$\int_{k/\xi}^{g_{\max}}\frac{dg}{g}\ln\frac{4kp}{\left(1-\dfrac{p}{k}\right)\left[g^2+\dfrac{1}{2}\dfrac{k}{p}\dfrac{(mc^2)^2}{(1-p/k)}\right]}\sim\begin{cases}\ln^2\dfrac{2\xi}{(k/p-1)^{\frac{1}{2}}}, & \dfrac{k}{\xi}<\kappa=\left[\dfrac{1}{2}\dfrac{k}{p}\dfrac{(mc^2)^2}{(1-p/k)}\right]^{\frac{1}{2}},\\ \ln\dfrac{2^{\frac{1}{2}}p}{mc^2}\ln\dfrac{2^{\frac{1}{2}}\xi^2}{(1-p/k)}\dfrac{mc^2}{k}, & \dfrac{k}{\xi}>\kappa.\end{cases}$$

[11] In the order in which we are working the introduction of a nuclear radius $\sim R$ within which the fields (11) are incorrect will not affect our results.

In carrying out the integrations over p and q we may again in the argument of the logarithm put $p\sim q\sim\frac{1}{2}k$, and set $k\sim k_0=2q_0$, where q_0 is the value of q at the maximum of $q|f(q,q)|^2$. The dominant terms in the cross section are

$$\begin{aligned}\phi &=\phi_0 \ln^2 \xi, && \xi<k_0/mc^2,\\ \phi &=\phi_0 \ln (k_0/mc^2) \ln (\xi^2 mc^2/k_0), && \xi>k_0/mc^2,\\ \phi_0 &=\frac{G^2Z^2r_0^2}{4\pi^3(mc^2)^2}\int_0^\infty q|f(q,q)|^2dq. &&\end{aligned}\tag{21}$$

We can also calculate, from (14), the energy distribution of the emitted positrons in the coordinate system in which the nucleus is at rest. If in the coordinate system in which the proton is initially at rest the positron is moving at an angle θ to the z axis, with energy ϵ and velocity u, it has in the rest system of the nucleus the energy

$$E=\xi\epsilon[1-(uv/c^2)\cos\theta]\sim\xi p(1-\cos\theta)=\xi px.$$

The energy distribution is obtained by introducing E as a new variable in place of x in (14), and reversing the order of integrations, leaving the integration over E till last. The cross section for emission of a positron in the energy range dE at E is thus

$$\phi(E)dE=\frac{G^2Z^2r_0^2}{2\pi^3(mc^2)^2}\frac{dE}{\xi}\int_0^\infty q^2|f(q,q)|^2dq\int_{E/2\xi}^\infty\frac{dp}{p+q}\int_{k/\xi}^\infty\frac{dg}{g}\frac{1}{\{[g^2+(k/p)(mc^2)^2+2kE/\xi]^2-8g^2pE/\xi\}^{\frac{1}{2}}}.$$

The integral over g gives approximately

$$\frac{\xi}{4kE}\ln\frac{2\xi E}{k(1-p/k)},\quad E>\frac{(mc^2)^2\xi}{2p},\ \frac{k}{\xi},$$

and the cross section is found to be

$$\phi(E)dE=\tfrac{1}{2}\phi_0\frac{dE}{E}\ln\frac{\xi E}{k_0},\qquad \xi k_0>E>\frac{(mc^2)^2\xi}{k_0},\ \frac{k_0}{\xi}.\tag{22}$$

For $E>\xi k_0$ the distribution falls off rapidly with E, the exact form of the high energy tail depending on the amplitude function f.[12]

For the total energy transmitted to the positrons per cm path, we obtain

$$W_e=N\int\phi(E)EdE\sim N\xi k_0\phi_0\ln\xi,\tag{23}$$

where N is the number of atoms per cm^3.

The angular distribution of the neutrinos in the rest system of the proton will be uniform for any amplitude function which does not contain a constraint between the direction of emission of the positron and neutrino. The energy lost to neutrinos per cm path is thus

$$W_\nu=N\xi\bar{q}\phi,\qquad \bar{q}=\int_0^\infty q^2|f(q,q)|^2dq\Big/\int_0^\infty q|f(q,q)|^2dq.\tag{24}$$

This is larger than W_e, because the positrons are ejected preferentially in the forward direction in the rest system of the proton, and therefore receive less energy. It is evident that the introduction of a constraint between the directions of the positron and neutrino could reduce W_ν to a value comparable to W_e.

4. Results and Discussion

Our final results for the cross section, energy distribution and energy losses for proton disintegration are contained in the expressions (21) to (24). We have now to evaluate ϕ_0, using the amplitude functions $f(p,q)$ discussed in §2.

[12] In calculating the high energy tail it is not legitimate to replace $f(p',q)$ by $f(q,q)$. With the amplitude function (7), the cross section for high energies falls off exponentially.

From (7) we obtain

$$\phi_0=\frac{G^2Z^2r_0^2}{4\pi^3(mc^2)^{2s+2}}\int_0^\infty q^{2s+1}e^{-4q/A}dq$$

$$=\frac{Z^2r_0^2mc^2}{k_0}\frac{m}{M}\frac{4\pi(2s+1)}{T_s(2s+2)}=Z^2r_0^2\frac{mc^2}{k_0}t_s, \qquad (25)$$

with $k_0=\frac{1}{2}(2s+1)A$, and the value of G^2 given in (8). Taking the numerical values for T_s from §2,

$$t_s=6.6\times10^{-3} \quad \text{for} \quad s=2, \quad n=1,$$

$$t_s=8.8\times10^{-3} \quad \text{for} \quad s=4, \quad n=2.$$

For $\bar{q}$, which appears in (24), we find

$$\bar{q}=(s+\tfrac{1}{2}s+1)k_0.$$

With $s=4$, $n=2$, $k_0\sim250\ mc^2$.

The Gaussian amplitude function (9) gives

$$\phi_0=\frac{G^2Z^2r_0^2}{2\pi^3(mc^2)^3}\int_0^\infty q^2e^{-2(q-p_0)^2/\Delta^2}dq$$

$$=(2\pi)^{\frac{1}{2}}Z^2r_0^2(mc^2/\Delta)(m/M) \qquad (26)$$

for $\Delta\ll p_0$, with the help of (10). With the correspondence $\Delta\sim k_0/(2s)^{\frac{1}{2}}$, (26) gives practically the same results as (25). We see from (25) and (26) that the cross section does not depend sensitively on the details of the amplitude function.

The cross section is of the order of magnitude $2\pi r_0^2(Z^2/137)(m/M)\ln^2\xi$ and is therefore small even for large Z. As the main contribution comes from impact parameters of the order $\rho\leq\xi\hbar c/k_0\sim\frac{1}{2}\xi r_0$ the probability of a disintegration for a single passage is very small, which is a necessary condition for the applicability of the Born approximation. The energy given to light particles per impact is, however, quite large, the mean energy lost to the positron being $\sim k_0\xi/\ln\xi$, or about $\frac{1}{2}$ billion volts for a 10 billion volt proton.

To obtain an idea of the total energy transmitted to the light particles, we compare (23) and (24) with the ordinary ionization losses which are given by

$$W_{\text{ion}}=2\pi NZr_0^2mc^2\ln(2\xi^3m^2c^4/Ry^2Z^2),$$

where $Ry=27$ ev.

TABLE I. *Numerical values for the energies transmitted to positrons and neutrinos, and the loss by ionization for a number of different energies in air (expressed in water equivalent) and in lead.*

	ξ	W_e Mev/cm	W_ν Mev/cm	W_{ion} Mev/cm	W_{ion} $E>100$ Mev Mev/cm	Distance per disintegration meters
Air (water equivalent)	10	0.017	0.022	1.7	0	320
	10^2	0.35	0.88	2.2	0.27	80
	10^3	5.2	19	2.7	0.61	36
Pb	10	1.8	2.3	12	0	3.0
	10^2	36	90	18	2.4	0.75
	10^3	540	2000	22	5.5	0.35

In Table I we give numerical values for the energies transmitted to positrons and neutrinos, and the loss by ionization, for a number of different energies in air (expressed in water equivalent) and in lead. The figures given are for $s=4$, $n=2$, but depend little on the particular values chosen.

We have already mentioned that the model which we have assumed for the Fermi coupling precludes its importance for cosmic ray showers. If, however, it be supposed that heavy particles constitute an essential part of the penetrating component of cosmic rays, then the proton (and neutron) disintegrations considered in this paper can provide a mechanism for transferring a considerable fraction of the proton's energy to a light particle, and thus for initiating showers at depths in matter to which no electrons or photons can penetrate. For in most of these disintegrations the positron will have a high enough energy to initiate a small shower.[13] In order to compare the effectiveness of these disintegrations with that of the energy transfer by direct impact with extranuclear electrons, we give in column 4 the energy lost per cm by such impacts to electrons of energy greater than 100 Mev.

In the last column of Table I we give the mean distance traveled by a proton per disintegration. This distance is of the right order of magnitude to account for the observed incidence of showers under great thicknesses (>10 cm) of Pb.

[13] A discussion of shower production by high energy electrons has been given by J. F. Carlson and J. R. Oppenheimer, Phys. Rev. **51**, 220 (1937) and by H. T. Bhabha and W. Heitler, Proc. Roy. Soc. **A159**, 432 (1937).